VPM CLASSES

IIT JAM

MATHEMATICAL STATISTICS

SOLVED SAMPLE PAPER

* DETAILED SOLUTIONS
* PROJECTED IIT JAM RANK

9001894070

www.vpmclasses.com

IIT JAM-MS MATHEMATICAL STATISTICS (FMTP)

Attempt ALL the 60 questions.

There are a total of 60 questions carrying 100 marks.

Section-A contains a total of 30 **Multiple Choice Questions (MCQ).** Q.1 - Q.10 carry 1 mark each and Questi ons Q.11 - Q.30 carry 2 marks each.

Section-B contains a total of 10 **Multiple Select Questions (MSQ).** Questions Q.31 - Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

Section-C contains a total of 20 **Numerical Answer Type (NAT)** questions. Questions Q.41 - Q.60 belong to this section and carry a total of 30 marks. Q.41 - Q.50 carry 1 mark each and Questions Q.51 - Q.60 carry 2 marks each.

In **Section-A** for all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section-B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section-C** (NAT) as well.

Time : 3 Hours MAX.MARKS : 100 MARKS SCORED :

SECTION-A (Q. 1-30): MULTIPLE CHOICE QUESTIONs (MCQs)

1. Let $P = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 5 & 4 & 2 \\ 3 & 6 & 6 & 4 \\ 1 & 8 & 3 & 0 \end{bmatrix}$

Then rank of P equals

(A) 4 (B) 3 (C) 2 (D) 1

2. Which of the following is a linear transformation?

(A) $T : R^3 \to R^2$ defined by $T(x, y, z) = (x, z)$

(B) $T : R^3 \to R^2$ defined by $T(x, y, z) = (x + y, 2z + 1)$

(C) $T : R^3 \to R^2$ defined by $T(x, y, z) = (x - y, z)$

(D) $T : R^3 \to R^2$ defined by $T(x, y, z) = (y, x)$

3. If a sequence $\{x_n\}$ is monotone and bounded, then

(A) There exists a subsequence of $\{x_n\}$ that diverges

(B) There may exist a subsequence of $\{x_n\}$ that is not monotone

(C) All subsequences of $\{x_n\}$ converge to the same limit

(D) There exists atleast two subsequences of $\{x_n\}$ which converges to distinct limits.

4. Let $f : R \to R$ be defined by $f(x) = x(x^2 - 1)$, then

(A) f is neither one-one nor onto. (B) f is one-one but not onto.

(C) f is onto but not one-one. (D) f is one-one and onto.

5. Let a_n be a sequence of positive real numbers such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{4}$. Then $\lim\limits_{n\to\infty}\frac{e^{a_n^2} + a_n^3}{\log(e + a_n)}$ equals

(A) ∞ (B) $\left(\frac{e^{1/4}}{4} + \frac{1}{8}\right)\frac{1}{\log 2}$ (C) $\frac{e^{1/16} + \frac{1}{64}}{\log\left(e + \frac{1}{4}\right)}$ (D) 1

6. Let E and F be two events with P(E) = 0.6; P(F) = 0.5; $P(E \cap F^C) = 0.4$. Then $P(F \mid E \cup F^C)$ is equal to,

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{1}{16}$

7. Let x be a continuous random variable with pdf

$$f(x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \; x \in R$$

Then $E(x^2)$ equals

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) does not exist

8. Let x_1, x_2, x_3 be a random sample from a distribution with a given pdf and a parameter ($\theta > 0$). If the expectation of the distribution is θ, then which of the following estimator for θ has the smallest variance?

(A) $\frac{x_1 + 2x_2 + x_3}{4}$ (B) $\frac{x_1 + x_2 + 3x_3}{5}$ (C) $\frac{x_1 + 2x_2 + 3x_3}{6}$ (D) $\frac{2x_1 + 3x_2 + 4x_3}{9}$

9. Let the random variable x have uniform distribution on the interval $\left(0, \frac{\pi}{3}\right)$. Then $P(\cos x > \sin x)$ is

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{2}{4}$ (D) $\frac{3}{4}$

10. Let $x_n \sim Exp(n)$. Then the sequence x_n converges in probability to

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) 0 (D) can not be found.

11. Player P_1 tosses 5 fair coins and P_2 tosses a fair die independently of P_1. The probability that the number of heads observed is more than the number on the upperface of the die, equals

(A) $\frac{7}{9}$ (B) $\frac{79}{96}$ (C) $\frac{9}{73}$ (D) $\frac{7}{73}$

12. Let x_1, x_2 and x_3 be i.i.d.r.vs with pmf

$$P(K) = \left(\frac{4}{5}\right)^{K-1}\left(\frac{1}{5}\right); \qquad K = 1, 2, 3, ...$$

Let $y = x_1 + x_2 + x_3$. Then $P(y \geq 5)$ equals

(A) $\frac{17}{625}$ (B) $\frac{608}{625}$ (C) $\frac{17}{125}$ (D) $\frac{108}{125}$

13. Let x and y be continuous random variables with the joint pdf

$$f(x, y) = \begin{cases} Cx^2(1-x), & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

where C is a positive real constant. Then E(x) equals

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

14. Let x and y be continous random variables with the joint pdf

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $P\left(x + y > \frac{1}{4}\right)$ equals

(A) $\frac{1}{192}$ (B) $\frac{95}{96}$ (C) $\frac{191}{192}$ (D) $\frac{1}{96}$

15. Let x be a standard normal random variable. Then $P(x^2 - 5x + 6 > 0)$ is

(A) $\Phi(2) + \Phi(3)$ (B) $\Phi(2) + \Phi(-3)$

(C) $\Phi(-2) + \Phi(3)$ (D) $\Phi(-2) + \Phi(-3)$

16. Let x, y be random variables having the joint pdf given by

$$f(x, y) = \begin{cases} \frac{1}{y}, & \text{if } 0 < y \le 1, 0 < x \le y \\ 0, & \text{otherwise.} \end{cases}$$

Then the corelation coefficient of x, y is given by

(A) $\sqrt{\frac{7}{12}}$ (B) $\sqrt{\frac{12}{7}}$ (C) $\sqrt{\frac{12}{5}}$ (D) $\sqrt{\frac{5}{12}}$

17. Let $x_1, x_2, ..., x_n$ be i.i.d. with following PMF

$$P_x(k) = \begin{cases} 0.75 & k = 2 \\ 0.25 & k = -2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $y = \sum_{i=1}^{n} x_i$, then an estimate of $P(0 \le y \le 2n)$ using CLT is

(A) $\Phi(0.34)$ (B) $1 - 2\Phi(0.34)$

(C) $2\Phi(0.34) - 1$ (D) $\Phi(0.34) + \Phi(0.66)$

18. Let $x_1, x_2, x_3, ..., x_5$ be a random sample from a geometric distribution with parameter θ, unknown. Then find the MLE of θ based on the random sample. Given that $x_k = k^2$ $1 \le k \le 5$ is

(A) 0.1 (B) 0.091 (C) 0.111 (D) 0.222

19. A random sample $x_1, x_2, ..., x_{121}$ is given from a distribution with known variance $var(x_i) = 25$. Let the sample mean by $\bar{x} = 12.83$. Then the upper limit of a 99% confidence interval for $\theta = E(x)$ is [Assume $\Phi^{-1}(0.995) = 2.574$]

(A) 14 (B) 15 (C) 16 (D) 18

20. Let $x_1, x_2, ..., x_n$ be a random sample from a Geometric distribution with parameter $\theta(> 0)$, where $\theta > 0$ is unknown. The Cramer-Rao lower bound for the variance of any unbiased estimator of θ is

(A) $\frac{\theta^2}{n}$ (B) $\frac{1-\theta}{n}$ (C) $\frac{\theta^2(1-\theta)}{n}$ (D) $\frac{\theta}{n}$

21. If $\int_{y=0}^{1}\int_{x=y}^{2-\sqrt{1-(y-1)^2}} f(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{\alpha(x)} f(x,y)dydx + \int_{x=1}^{2}\int_{y=0}^{\beta(x)} f(x,y)dydx$

then $[\beta(x) - 1]^2 + [\alpha(x) - 2]^2$ equals

(A) x (B) 1 (C) $1-\sqrt{1-(x-2)^2}$ (D) x – 1

22. Let f : [0, 1] → R be a function defined as

$$f(t) = \begin{cases} \ln(1+2t).\dfrac{\sin t^3}{t^2}, & t \neq 0 \\ 2, & \text{otherwise.} \end{cases}$$

Let F : [0, 1] → R be defined as

$$F(x) = \int_0^x f(t)dt$$

Then F"(0) equals

(A) 0 (B) 2 (C) $\frac{1}{2}$ (D) –2

23. Let x, y and z are real numbers such that 2x + 3y + 4z = 12 and x + 3y – z = 15. Then the value of x + 2y + z equals.

(A) cannot be computed from the given information.

(B) equals 9

(C) equals $\frac{26}{3}$

(D) equals $\frac{25}{3}$

24. Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$. If

$$V = \left\{(x,y,0) \in R^3 : M\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$ then the dimension of V is

(A) 0 (B) 1 (C) 2 (D) 3

25. Let M be a 3 × 3 non-zero, skew-Hermitian real matrix, I is the 3 × 3 identity matrix then,

(A) M is invertible.

(B) the matrix I + M is invertible.

(C) $\alpha I + M$ is not invertible for some $\alpha \in R\backslash\{0\}$.

(D) Eigen values of M are imaginary.

26. Let the sequence $\{u_n\}_{n \geq 1}$ be defined by

$$u_n = \frac{\left(1+\frac{C}{n}\right)^{n^2}}{\left(5-\frac{2}{n}\right)^n}, C \in R.$$

Then the values of C for which the series $\sum_{n=1}^{\infty} u_n$ converges are

(A) $\log_e 5 < C < \log_e 10$

(B) $C > \log_e 10$

(C) $C < \log_e 5$

(D) $0 < C < \log_e 10$

27. If for a suitable $\alpha > 0$,

$$\lim_{x \to 0}\left(\frac{1}{\ln(1+5x)} - \frac{1}{\alpha x}\right)$$

exists and is equal to $\ell(|\ell| < \infty)$, then

(A) $\alpha = 5, \ell = 2$

(B) $\alpha = \frac{1}{5}, \ell = \frac{1}{2}$

(C) $\alpha = \frac{1}{5}, \ell = 5$

(D) $\alpha = 5, \ell = \frac{1}{2}$

28. Let Q, A, B be matrices of order n × n real entries such that Q is orthogonal and A is invertible. Then the eigenvalues of $Q^TA^{-1}BQ$ are always the same as those of

(A) AB

(B) $Q^TA^{-1}B$

(C) $A^{-1}BQ^T$

(D) BA^{-1}

29. Let y(x) be the solution to the differential equation

$$x^4\frac{dy}{dx} + 4x^3y - \cos x = 0,\ y(\pi) = 1,\ x > 0$$

Then $y\left(\frac{\pi}{2}\right)$ is

(A) $\frac{16(1+\pi^4)}{\pi^4}$ (B) $\frac{14(1+\pi^4)}{\pi^4}$ (C) $\frac{12(1+\pi^4)}{\pi^4}$ (D) $\frac{10(1+\pi^4)}{\pi^4}$

30. Let y(x) be the solution to the differential equation

$$4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0,\ y(0) = 1,\ y'(0) = -4,$$

Then y(1) equals

(A) $-\frac{1}{2}e^{-5/2}$ (B) $-\frac{3}{2}e^{-5/2}$ (C) $-\frac{5}{2}e^{-5/2}$ (D) $-\frac{7}{2}e^{-5/2}$

SECTION-B (Q. 31-40): MULTIPLE SELECT QUESTIONs (MSQs)

31. The differential equations that satisfy $y_1 = e^{-x}\sin(x)$, $y_2 = e^{-3x}$ are

(A) $\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}+8\frac{dy}{dx}+6y=0$ (B) $\frac{d^3y}{dx^3}+8\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

(C) $\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}-2\frac{dy}{dx}-6y=0$ (D) $\frac{d^4y}{dx^4}-4\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}-2\frac{dy}{dx}+6y=0$

32. Which of the following are true with respect to matrices

(A) A skew-symmetric matrix can be orthogonal.

(B) A symmetric matrix can be orthogonal.

(C) An orthogonal matrix can have a 0 eigen value.

(D) A symmetric matrix can have the expression $x^3 + x^2 - 2$ as characteristic expression.

33. Which of the following are true in a probability space (Ω, F, P)

(A) $P(A) = 0 \Rightarrow A = \phi$

(B) A and B are independent events $\Rightarrow A \cap B = \phi$

(C) $A = \Omega \Rightarrow P(A) = 1$

(D) $A \subset B \Rightarrow P(A \mid B) = P(A)/P(B)$

34. Let x be a random variable such that $2 \le x \le 6$. Then an upper limit for Var(x) is

(A) 1 (B) 4 (C) 8 (D) 16

35. If $x_1, x_2, x_3, ..., x_n$ be a random sample, then

(A) $\bar{x}$ is an unbiased estimator of $\theta = E(x)$

(B) x_1 is an unbiased estimator of $\theta = E(x)$

(C) $\bar{x}$ is more efficient among $\bar{x}$ and $\frac{x_1 + x_2}{2}$ for $\theta = E(x)$

(D) $\bar{x}$ is a consistent estimator of $\theta = E(x)$

36. Let $f : R \to R$, $g : R \to R$, then which of the following is/are true.

(A) g(x) = sin f(x) + sin f(–x) is even for all f.

(B) For all odd functions f, g(x) = sin f(x) + sin f(–x) is an even function.

(C) g(x) = f(x) [f(x) + f(–x)] is even if f is even.

(D) g(x) = f(sinx) + f(cosx) is odd if f is even.

37. The function $y = x(x - 1)^2$, $0 \le x \le 2$, has

(A) y has a global maximum at $x = \frac{1}{3}$. (B) y has a global minimum at x = 1.

(C) y has a local maximum at $x = \frac{1}{3}$. (D) y has a local minimum at x = 1.

38. Let the pdf of a random variable x be given by

$f(x) = C_1 x^{C_2} (1 - x)^{C_3} x_{(0,1)}(x), -\infty < x < \infty$

If $E(x) = \frac{2}{5}$ and $Var(x) = \frac{6}{275}$, then

(A) $C_1 = 105$ (B) $C_2 = 3$

(C) $C_3 = 4$ (D) $C_3 = 5$

39. Let x_1, x_2, x_3, x_4 be random samples of size 4 from a standard normal distribution. Then

(A) $x_1^2 \sim x^2(1)$ (B) $\frac{x_1}{\sqrt{x_2^2}} \sim t(1)$

(C) $\frac{x_1 - x_2 + x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}$ has expectation 0 (D) $x_1 - x_2 + x_3 \sim N(0, 1)$

40. Let x be a random variable with mean 2. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be unbiased estimators of the second and third moments respectively of x about origin. Then

(A) Unbiased estimator for second moment about mean is $\hat{\theta}_1 - 4$.

(B) $\hat{\theta}_1 + 4$ is an unbiased estimator for second moment about mean.

(C) Unbiased estimator for third moment about mean is $\hat{\theta}_2 - 6\hat{\theta}_1 + 16$.

(D) $\hat{\theta}_2 - 6\hat{\theta}_1$ is an estimator of third moment about mean.

SECTION-C (Q. 41-60): NUMERICAL ANSWER TYPE (NATs)

41. Let the random variable x have uniform distribution on the interval (0, 1) and $y = e^{-x}$. Then E(y) equals ______.

42. If $y = \log_{10}x$ has $N(\mu, \sigma^2)$ distribution with MGF $M_y(t) = e^{2t+2t^2}$, $t \in (-\infty, \infty)$, then $P(x < 100)$ equals ____.

43. Two events A and B have probabilities 0.26 and 0.54 respectively. The probability that both A and B occur simultaneously is 0.2. The probability that neither A nor B occur is ___.

44. Consider the linear transformation

$$T(x, y, z) = (x + y + z, 2x + y - 3z, 3x + 2y - 2z)$$

Then rank of T is ______.

45. Consider the differentiable function $f : R \to R$ with derivative $f'(x) = \frac{2\tan x}{1 - \tan^2 x}$.

The arc length of the curve $y = f(x)$, $\frac{\pi}{12} \le x \le \frac{\pi}{6}$ is _____.

46. Let x and y be discrete random variables with joint pmf

$$P(x, y) = \frac{1}{25}(x^2 + y^2) \quad \text{if } x = 1, 2;\ y = 0, 1, 2.$$

Then $P(y = 0 \mid x = 2)$ equals ____.

47. The area bounded by the parabola $y = x^2 + 4$, $y = 0$, $x = 0$ and $x = 3$ is ____.

48. Let x be a continuous random variable with the pdf

$$f(x) = \begin{cases} \dfrac{3x^2}{125}, & 0 < x < 5 \\ 0, & \text{otherwise.} \end{cases}$$

Then the upper bound of $P\left(|x-3| > \sqrt{3}\right)$ using chebyshev's inequality is ______.

49. Let [x] be the greatest integer less than or equal to x. Then $\int_{-2}^{1} [x]^2 dx$ = _______.

50. Let x be a random variable with density $f(x) = \frac{1}{4} e^{-|x|/2}$, $-\infty < x < \infty$. Then $E(|x|^3)$ = ____.

51. Based on 10 observations (x_i, y_i); i = 1, 2, ..., 10, the following are obtained. $\sum_{i=1}^{10} x_i = 30, \sum_{i=1}^{10} y_i = 50, \sum_{i=1}^{10} x_i y_i = 160, \sum_{i=1}^{10} x_i^2 = 110$. For x = 5, the predicted value of y based on a least squares fit of a linear regression model of y on x is ____.

52. Let $P = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$. Then the trace of P^{-1} is _____.

53. The value of $\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$ equals _____.

54. Let $x_1, x_2, ..., x_8$ be a random sample from a population with true mean μ and variance σ^2. Let two estimators of θ be

$$\hat{\theta}_1 = \frac{x_1 + x_8}{2}$$

$$\hat{\theta}_2 = \frac{x_1 + x_8}{4} + \frac{x_2 + x_3 + ... + x_7}{12}$$

Then the relative efficiency $e(\hat{\theta}_2, \hat{\theta}_1)$ equals ______.

55. Let $x_1, x_2, ..., x_{20}$ be a random sample from a N(12.37, 99) population. Suppose $y_1 = \frac{1}{11}\sum_{i=1}^{11} x_i$ and $y_2 = \frac{1}{9}\sum_{i=12}^{20} x_i$. If $\frac{(y_1 - y_2)^2}{\alpha}$ has a x_1^2 distribution, then α = ______.

56. Let x be a single observation taken from a normal population with unknown mean μ and known variance 0.2. The fischer information in x about μ is _____.

57. The probability density function of a random variable x is given by

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } |x| \le 1 \\ \frac{1}{4x^2}, & \text{otherwise.} \end{cases}$$

then $P(0 \le x \le 4)$ = ________.

58. Let a unit vector $v = (v_1 \quad v_2 \quad v_3)^T$ be such that Av = 0 where

$$A = \begin{pmatrix} 5 & -2 & -1 \\ -1 & 1 & -1 \\ -1 & -2 & 5 \end{pmatrix}$$

then $\frac{|v_2|^2 - |v_3|^2}{|v_1^2|}$ equals ______.

59. A tangent is drawn on the curve $y = \frac{2}{5}\sqrt{x^5}$, (x > 0) at the point $P\left(1, \frac{2}{5}\right)$ which meets the x-axis at Q. Then the length of the closed curve OQPO, where O is the origin, is ____.

60. Let y(x) be the solution of the differential equation

$$x^5 \frac{dy}{dx} + 5x^4 y + e^{-x} = 0$$

satisfying the condition y(1) = 0, then $\frac{1}{e^2}\frac{y(-1)}{y(3)}$ = ______.

ANSWER KEY FMTP

Ques	1	2	3	4	5	6	7	8	9	10
Ans	A	B	C	C	D	C	B	A	D	C
Ques	11	12	13	14	15	16	17	18	19	20
Ans	B	B	D	C	B	B	C	B	A	C
Ques	21	22	23	24	25	26	27	28	29	30
Ans	B	A	B	A	B	C	D	D	A	A
Ques	31	32	33	34	35	36	37	38	39	40
Ans	A,C	A,B	C,D	B,C,D	A,B,C,D	A,B,C,D	B,C,D	A,B,D	A,B,C	A,C,D
Ques	41	42	43	44	45	46	47	48	49	50
Ans	0.63	0.5	0.4	2	0.167	0.24	21	2	5	48
Ques	51	52	53	54	55	56	57	58	59	60
Ans	6	1.4	0.5	3	20	5	0.4375	3	2.24	243

HINTS & SOLUTION

Sol.(1) (A) $$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 5 & 4 & 2 \\ 3 & 6 & 6 & 4 \\ 1 & 8 & 3 & 0 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -3 & 8 \\ 0 & 6 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 6R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 12 & -23 \end{bmatrix} \xrightarrow{R_4 \to R_4 + 4R_3} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$\therefore$ rank of P = 4

Sol.(2) (B) For option B, $T(u + v) = T(u_1 + v_1, u_2 + v_2, u_3 + v_3)$

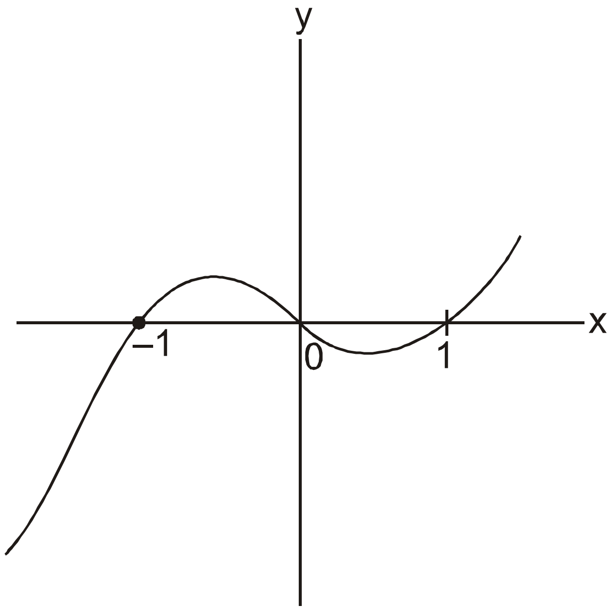
$= (u_1 + v_1 + u_2 + v_2, 2u_3 + 2v_3 + 1)$

$Tu + Tv = (u_1 + v_2, 2u_3 + 1) + (v_1 + v_2, 2v_3 + 1) = (u_1 + v_1 + u_2 + v_2, 2u_3 + 2v_3 + 2)$

$\neq T(u + v)$

Sol.(3) (C) By the Bolzano-Weistrass theorem all subsequences of a monotone bounded sequence converge to the same limit.

Sol.(4) (C)



The function between [–1, 1] is not one-one. It takes 0 at 3 different points.

So, its onto but not one-one.

Sol.(5) (D) $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{4}$ $\Rightarrow \lim\limits_{n\to\infty}a_n=0$

$$\Rightarrow \lim_{n\to\infty}\frac{e^{a_n^2}+a_n^3}{\log(e+a_n)} = \frac{e^{0^2}+0^3}{\log(e+0)}=\frac{1}{1} = 1$$

Sol.(6) (C) $P(E \cap (E \cup F^C)) = P((F \cap E) \cup (F \cap F^C)) = P(F \cap E) = P(E) – P(E \cap F^C)$

$= 0.5 – 0.4 = 0.1$

$$\therefore P(F \mid E \cup F^C) = \frac{P(F\cap(E\cup F^C))}{P(E\cup F^C)}=\frac{0.1}{0.6+0.6-0.4} = \frac{1}{8}$$

Sol.(7) (B) Normal distribution pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}}.e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Comparing we get

$\mu = 0,$ $2\sigma^2 = 1$

$$\Rightarrow \sigma = \frac{1}{\sqrt{2}}$$

$\sigma^2 = E(x^2) – E(x)^2$

$$\Rightarrow E(x^2) = \sigma^2 + \mu^2 = \frac{1}{2} + 0 = \frac{1}{2}$$

Sol.(8) (A) Let $Var(x) = \sigma^2$

$$Var\left(\frac{x_1+2x_2+x_3}{4}\right)=\sigma^2\left(\frac{1+4+1}{16}\right)=\frac{3\sigma^2}{8} = 0.375\sigma^2$$

$$Var\left(\frac{x_1+x_2+3x_3}{5}\right)=\frac{11\sigma^2}{25} = 0.44\sigma^2$$

$$Var\left(\frac{x_1+2x_2+3x_3}{6}\right)=\frac{7\sigma^2}{18} = 0.389\sigma^2$$

$$Var\left(\frac{2x_1+3x_2+4x_3}{9}\right)=\frac{29\sigma^2}{81} = 0.358\sigma^2$$

Sol.(9) (D) $P(\cos x > \sin x) = P(\tan x < 1) = P\left(0<x<\frac{\pi}{4}\right)=\frac{\pi/4-0}{\pi/3-0} = \frac{3}{4}$

Sol.(10) (C) $\lim\limits_{n\to\infty}P\left(|x_n-0|\ge\varepsilon\right)=\lim\limits_{n\to\infty}P(x_n\ge\varepsilon)=\lim\limits_{n\to\infty}e^{-n\varepsilon} = 0$

Sol.(11) (B) 5 heads $\rightarrow$ numbers : 1, 2, 3, 4 $\quad P = \left(\frac{1}{2}\right)^5\times\frac{4}{6}=\frac{1}{8}\times\frac{1}{3}=\frac{1}{24}$

4 heads $\rightarrow$ numbers : 1, 2, 3 $\quad P = \left(\frac{1}{2}\right)^4\times5\times\frac{3}{6}=\frac{5}{32}$

3 heads $\rightarrow$ numbers : 1, 2 $\quad P = \left(\frac{1}{2}\right)^3\times10\times\frac{2}{6}=\frac{5}{12}$

2 heads $\rightarrow$ numbers : 1 $\quad P = \left(\frac{1}{2}\right)^2\times5\times\frac{1}{6}=\frac{5}{24}$

$\therefore P_{tot} = \frac{79}{96}$

Sol.(12) (B) $P(y \ge 5) = 1 - P(y < 5) = 1 - P(x_1 + x_2 + x_3 < 5) = 1 - P(x_1 + x_2 + x_3 \le 4)$

$= 1 - [P(x_1 = 1, x_2 = 1, x_3 = 1) + 3P(x_1 = 1, x_2 = 1, x_3 = 2)]$

$$= 1-\left[\left(\frac{1}{5}\right)^3+3\left(\frac{1}{5}\right)^3.\frac{4}{5}\right]=1-\left[\left(\frac{1}{5}\right)^3.\frac{17}{5}\right]=1-\frac{17}{625} = \frac{608}{625}$$

Sol.(13) (D) $\iint\limits_{R^2} f(x,y)dydx=1$

$$\Rightarrow \int_0^1\int_x^1 C.x^2(1-x)dydx=1$$

$$\Rightarrow C\int_0^1 x^2(1-x)^2dx=1$$

$$\Rightarrow C\left[\int_0^1 x^2dx-\int_0^1 2x^3dx+\int_0^1 x^4dx\right]=1$$

$$\Rightarrow C\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right]=1$$

$$\Rightarrow C\left[\frac{16-15}{30}\right]=1$$

$$\Rightarrow C = 30$$

$$E(x) = \iint xf(x,y)dydx = 30\int_0^1\left(\int_x^1 x^3(1-x)dy\right)dx = 30\int_0^1[x^3(1-x)^2]dx$$

$$=30\left[\int_0^1 x^3dx - 2\int_0^1 x^4dx + \int_0^1 x^5dx\right] = 30\left[\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right] = 30\left[\frac{1}{60}\right] = \frac{1}{2}$$

Sol.(14) (C) $P\left(x+y>\frac{1}{4}\right) = 1-P\left(x+y\le\frac{1}{4}\right)$

$$P\left(x+y\le\frac{1}{4}\right)=\int_0^{1/4}\int_0^{1/4-y}(x+y)dxdy = \int_0^{1/4}\left[\left(\frac{1}{4}-y\right)^2.\frac{1}{2}+y\left(\frac{1}{4}-y\right)\right]dy$$

$$=\frac{1}{2}\int_0^{1/4}\left[\left(\frac{1}{4}-y\right)^2+2y\left(\frac{1}{4}-y\right)\right]dy = \frac{1}{2}\left[\int_0^{1/4}\left(\frac{1}{16}-y^2\right)dy\right] = \frac{1}{2}\left[\frac{1}{16}.\frac{1}{4}-\frac{1}{3}.\left(\frac{1}{64}\right)\right]$$

$$=\frac{1}{2}.\frac{1}{64}.\frac{2}{3}=\frac{1}{192}$$

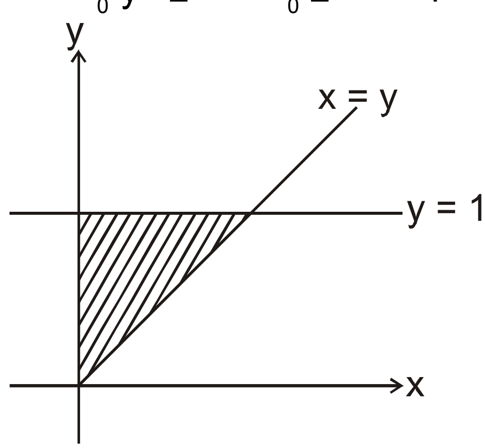
$$P\left(x+y>\frac{1}{4}\right) = 1-P\left(x+y\le\frac{1}{4}\right) = \frac{191}{192}$$

Sol.(15) (B) $P(x^2 - 5x + 6 > 0) = P[(x-2)(x-3) > 0] = 1 - P[(x-2)(x-3) \le 0]$

$= 1 - P(2 \le x \le 3) = 1 - [\Phi(3) - \Phi(-2)] = 1 - \Phi(3) + \Phi(2)] = 1 - \Phi(3) + \Phi(2)$

$= \Phi(-3) + \Phi(2)$

Sol.(16) (B) $E(x) = \int_0^1\int_0^y x.\frac{1}{y}dxdy = \int_0^1\frac{1}{y}.\frac{y^2}{2}dy = \int_0^1\frac{y}{2}dy = \frac{1}{4}$



$$E(y) = \int_0^1\int_x^1 y.\frac{1}{x}dydx = \frac{1}{2}$$

$$E(xy) = \int_0^1\int_0^y (xy).\frac{1}{y}dxdy = \frac{1}{6}$$

$$E(x^2) = \int_0^1\int_0^y x^2.\frac{1}{y}dxdy = \frac{1}{9}$$

$$E(y^2) = \int_0^1\int_0^y x^2.\frac{1}{x}dxdy = \int_0^1 (-x^2)dx = \frac{1}{3}$$

$$\text{cov}(x, y) = \frac{1}{6} - \frac{1}{2}.\frac{1}{4} = \frac{1}{12}$$

$$\text{Var}(x) = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}, \ \text{Var}(y) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore \rho(x, y) = \sqrt{\frac{12}{7}}$$

Sol.(17) (C) $E(x_i) = (0.75)(2) + (0.25)(-2) = (0.5)(2) = 1$

$E(x_i^2) = (0.75)(4) + (0.25)4 = 4$

$\therefore \text{var}(x_i) = 4 - 1 = 3$

$$E(y) = \sum_{i=1}^{n} E(x_i) = n$$

$$\text{var}(y) = \sum_{i=1}^{n} \text{var}(x_i) = 3n$$

$$P(0 \le y \le 2n) = P(-n \le y - n \le n) = P\left(-\frac{1}{3} \le \frac{y-n}{3} \le \frac{1}{3}\right) = 2\Phi(0.34) - 1$$

Sol.(18) (B) $X_i \sim \text{Geom}(\theta)$. Then

$(x; \theta) = (1 - \theta)^{x-1}\theta$

$$\Rightarrow L(x_1, x_2, ..., x_5; \theta) = (1-\theta)^{\sum_{i=1}^{5} x_i - 5}\theta^5$$

$$\ln L = \left(\sum_{i=1}^{5} x_i - 5\right)\ln(1 - \theta) + 5\ln\theta$$

$$\frac{1}{L}\frac{\partial L}{\partial\theta} = -\frac{\sum_{i=1}^{5} x_i - 5}{1-\theta} + \frac{5}{\theta}$$

$$\boxed{\sum_{i=1}^{5} x_i = \frac{5\times 6\times 11}{6} = 55}$$

$$= -\frac{50}{1-\theta} + \frac{5}{\theta}$$

$$\frac{\partial L}{\partial\theta} = 0 \Rightarrow \frac{50}{1-\theta} = \frac{5}{\theta} \qquad \Rightarrow 10\theta = 1 - \theta$$

$$\Rightarrow 11\theta = 1$$

$$\Rightarrow \theta = \frac{1}{11} = 0.091$$

Sol.(19) (A) The approximate $(1 - \alpha)100\%$ confidence interval is given by

$$\left[\bar{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right]$$

here $\alpha = 0.01$, $z_{\alpha/2} = z_{0.005} = \Phi^{-1}(0.995) = 2.574$

given $\sigma^2 = 25 \quad \Rightarrow \sigma = 5$

Upper limit of confidence interval is

$$\bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = 12.83 + 2.574 \times \frac{5}{11} = 12.83 + 1.17 = 14$$

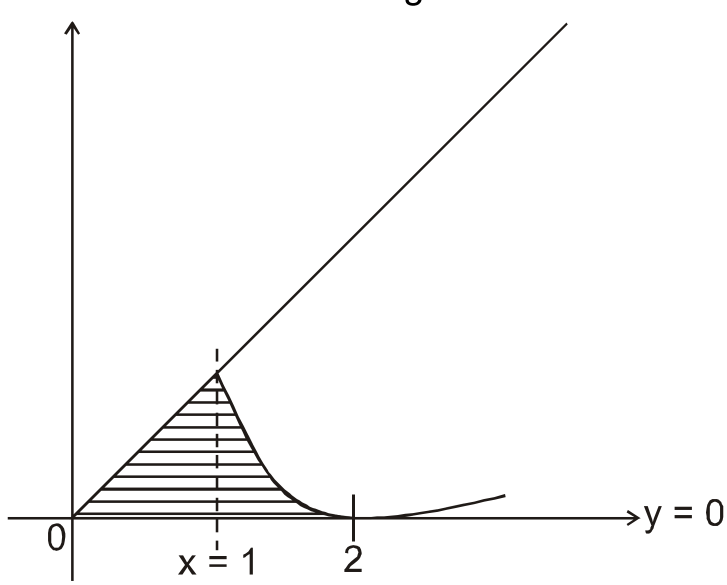
Sol.(20) (C) $P(x) = \begin{cases} \theta(1-\theta)^{x-1}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$

$$\frac{\partial}{\partial\theta}\ln P(x) = \frac{1}{\theta} + \frac{x-1}{\theta - 1}$$

$$\text{lower bound} = \frac{1}{nE\left[\left\{\frac{\partial \ln P(x)}{\partial\theta}\right\}^2\right]} = \frac{1}{n\text{Var}\left[\frac{\partial}{\partial\theta}\ln P(x)\right]} = \frac{(\theta-1)^2}{n\text{Var}(x-1)} = \frac{(\theta-1)^2}{n.\frac{(1-\theta)}{\theta^2}}$$

$$= \frac{\theta^2(1-\theta)}{n}$$

Sol.(21) (B) Change of variable order of integration



$x : 0 \to 1 \quad \Rightarrow y : 0 \to x$

$\therefore \alpha(x) = x$

$x : 1 \to 2 \quad \Rightarrow y : 0 \to 1+\sqrt{1-(x-2)^2}$

$\therefore\ \beta(x) = 1+\sqrt{1-(x-2)^2}$

$\therefore\ [\alpha(x) - 2]^2 + [\beta(x) - 1]^2 = 1$

Sol.(22) (A) F'(x) = f(x) by Fundamental theorem

$$F''(0) = f'(0) = \lim_{t\to 0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}\frac{\ln(1+2t).\frac{\sin t^3}{t^2}}{t} = \lim_{t\to 0}\ln(1+2t).\frac{\sin(t^3)}{t^3} = 0$$

Sol.(23) (B) $x + 2y + z = \frac{1}{3}[3x + 6y + 3z] = \frac{1}{3}[2x + 3y + 4z + x + 3y - z] = \frac{1}{3}[12 + 15]$

$= 9$

Sol.(24) (A) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} x+2y=0 \\ 8x+y=0 \end{matrix}\right\} \quad \Rightarrow x = 0,\ y = 0$

$\therefore \quad V = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad \dim(V) = 0$

Sol.(25) (B) A real Skew-Hermetian matrix is same as a skew symmetric matrix.

M is not invertible as eigenvalues must have 0.

So eigenvalues are not all imaginary.

$|\lambda I + M| = 0$ gives either $\lambda = 0$ or λ is imaginary.

$\therefore\ \nexists\ \alpha \in R\backslash\{0\}$ such that $|\alpha I + M| = 0$

$\therefore\ I + M$ is invertible.

Sol.(26) (C) Root test : $\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\frac{\left(1+\frac{C}{n}\right)^n}{5-\frac{2}{n}} = \frac{e^C}{5}$

Convergence $\Rightarrow \frac{e^C}{5} < 1 \quad \Rightarrow C < \log_e 5$

Sol.(27) (D) $\lim_{x\to 0}\left[\frac{\alpha x - \ln(1-5x)}{\ln(1+5x).\alpha x}\right] = \frac{1}{\alpha}\lim_{x\to 0}\left[\frac{\alpha - \frac{\ln(1+5x)}{x}}{\ln(1+5x)}\right]$

the denominator goes to 0, for the limit to exist, numerator should be 0.

$\alpha - \lim_{x\to 0}\frac{\ln(1+5x)}{x} = 0 \quad \Rightarrow \alpha - 5 = 0$

$\Rightarrow \alpha = 5$

$\Rightarrow \ell = \lim_{x\to 0}\left[\frac{5x - \ln(1+5x)}{5x\ln(1+5x)}\right] = \lim_{x\to 0}\left[\frac{5 - \frac{1}{1+5x}.5}{5\left[\ln(1+5x) + \frac{5x}{1+5x}\right]}\right]$

$= \lim_{x\to 0}\left[\frac{1}{1.1(1+5x)\frac{\ln(1+5x)}{5x}}\right] = \frac{1}{2}$

Sol.(28) (D) Similar matrices have same eigen values,

Q is orthogonal $\Rightarrow Q^T = Q^{-1}$

$\therefore$ $Q^TA^{-1}BQ$ is similar to $A^{-1}B$

$\therefore$ They have same eigen values,

AB and BA have same eigen values

$\Rightarrow$ $A^{-1}B$ and BA^{-1} have same eigen values.

Sol.(29) (A) $\frac{d}{dx}(x^4y) = \cos x$

$\Rightarrow x^4y = \sin x + C$

$\Rightarrow y = x^{-4}\sin x + Cx^{-4}$

$y(\pi) = 1 \quad \Rightarrow C = \pi^4$

$\therefore y(x) = \frac{1}{x^4}[x^4 + \sin x]$

$y\left(\frac{\pi}{2}\right) = \frac{16}{\pi^4}(\pi^4 + 1) = \frac{16(1+\pi^4)}{\pi^4}$

Sol.(30) (A) AE is $\quad 4m^2 + 20m + 25 = 0$

$\Rightarrow (2m + 5)^2 = 0$

$$\Rightarrow m = -\frac{5}{2}, -\frac{5}{2}$$

$$y(x) = C_1 e^{-\frac{5}{2}x} + C_2 x e^{-\frac{5}{2}x}$$

$y(0) = 1 \quad \Rightarrow C_1 = 1$

$y'(0) = -4 \quad \Rightarrow -\frac{5}{2}C_1 + C_2 = -4$

$$\Rightarrow C_2 = -4 + \frac{5}{2}(1) = -\frac{3}{2}$$

$$y(1) = C_1 e^{-\frac{5}{2}} + C_2 e^{-\frac{5}{2}} = e^{-\frac{5}{2}}(C_1 + C_2) = -\frac{1}{2}e^{-5/2}$$

Sol.(31) (A, C) $e^{-x}\sin(x)$ is a root $\Rightarrow e^{-x}\cos(x)$ is a root.

$\therefore$ –1 + i, –1 – i are roots of auxilliary equation.

$\therefore$ The auxilliary equation is

$$[m - (-1 - i)][m - (-1 + i)](m + 3) = 0$$

$$\Rightarrow \quad m^3 + 5m^2 + 8m + 6 = 0$$

$\therefore$ D.E. is $\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 6y = 0$

for option C, the Auxilliary equation is

$m^4 + 4m^3 + 3m^2 - 2m - 6 = (m - 1)(m^3 + 5m^2 + 8m + 6)$

$\therefore$ C is also correct.

Sol.(32) (A, B) A = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric and orthogonal.

A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric and orthogonal.

An orthogonal matrix has determinant ±1. So it must have all eigen values non-zero.

If $x^3 + x^2 - 2$ is characteristic expression, then eigen values are 1, –1 ± 2. A skew symmetric matrix cannot have a real eigen value.

Sol.(33) (C, D) For a continuous distribution, the probability of discrete points is 0.

Independent events $\Leftrightarrow P(A \cap B) = P(A).P(B)$

$A \cap B = \phi$ Þ $P(A \cap B) = 0$

but P(A).P(B) need not be 0.

$A = \Omega \quad \Rightarrow P(A) = 1$

$A \subset B \Rightarrow A \cap B = A \quad \Rightarrow P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)}{P(B)}$

Sol.(34) (B, C, D) Define y = x – 4,

$2 \le x \le 6 \quad \Rightarrow -2 \le y \le 2$

$\Rightarrow y^2 \le 4$

$Var(x) = Var(y)$

$= E(y^2) - [E(y)]^2$

$\le E(y^2)$

$\le E(4)$

$= 4$

$\therefore\ Var(x) \le 4$

Sol.(35) (A, B, C, D) $E(\bar{x}) = \dfrac{1}{n}\sum_{i=1}^{n} E(x_i) = E(x) = \theta$

$\therefore\ \bar{x}$ is unbiased.

$E(x_1) = E(x) = \theta$

$\therefore\ x_1$ is unbiased.

$MSE(\bar{x}) = E[\bar{x} - E(\bar{x})]^2 = Var[\bar{x} - E(x)] + E[\bar{x} - E(x)]^2 = Var(\bar{x}) = \dfrac{\sigma^2}{n}$

$MSE(x_1) = Var(x_1) = \sigma^2 > MSE(\bar{x})$

$\therefore\ \bar{x}$ is more efficient.

By chebyshev's inequality

$P[|\bar{x} - E(x)| \ge \varepsilon] \le \dfrac{Var(\bar{x})}{\varepsilon^2} = \dfrac{\sigma^2}{n\varepsilon^2}$

$\Rightarrow \lim_{n\to\infty} P\left[\left|\bar{x} - E(x)\right| \ge \varepsilon\right] = 0$

$\Rightarrow \bar{x}$ is consistent.

Sol.(36) (A, B, C, D) (A) g(–x) = sin f(–x) + sin f(x) = g(x)

$\therefore$ g is even for all f.

(B) f is odd $\Rightarrow$ f(–x) = –f(x)

$\Rightarrow$ g(x) = sin f(x) – sin f(x) = 0

$\therefore$ g is a constant $\Rightarrow$ g is both even and odd.

(C) g(–x) = f(–x)[f(–x) + f(x)] = f(x)[f(–x) + f(x)] = g(x) $\Rightarrow$ g(x) is even.

(D) g(–x) = f[sin(–x)] + f[cos(–x)] = f(sinx) + f(cosx) = g(x)

$\therefore$ g is even.

Sol.(37) (B, C, D) $\frac{dy}{dx} = (x-1)^2 + 2x(x-1) = (x-1)(3x-1)$

$\frac{dy}{dx} = 0 \quad \Rightarrow x = 1 \quad$ or $\quad x = \frac{1}{3}$

$\frac{d^2y}{dx^2} = 3x - 1 + x - 1 = 4x - 2$

$x = 1 \Rightarrow y'' > 0 \quad \Rightarrow$ y has a local minimum

$x = \frac{1}{3} \Rightarrow y'' < 0 \quad \Rightarrow$ y has a local maximum

at $x = 0$, $y(x) = 0$

at $x = \frac{1}{3}$, $y(x) = \frac{4}{27}$

at $x = 1$, $y(x) = 0 \Rightarrow x = 1$ has a global minimum.

at $x = 2$, $y(x) = 2$

Sol.(38) (A, B, D) The distribution given is of Beta.

$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \chi_{(0,1)}^{(x)}$

$E(x) = \frac{a}{a+b}$, $Var(x) = \frac{ab}{(a+b+1)(a+b)^2}$

here $C_1 = a - 1$, $C_2 = b - 1$

$E(x) = \frac{2}{5} \quad \Rightarrow \frac{a}{a+b} = \frac{2}{5} \quad \Rightarrow 3a = 2b$

$Var(x) = \frac{6}{275} \quad \Rightarrow \frac{ab}{(a+b+1)(a+b)^2} = \frac{6}{275}$

$\Rightarrow \frac{b}{(a+b+1)(a+b)} \cdot \frac{2}{5} = \frac{6}{275}$

$\Rightarrow \frac{b}{a(a+b+1)} \cdot \frac{4}{25} = \frac{6}{275}$

$\Rightarrow \frac{1}{a+b+1} \cdot \frac{3}{2} \cdot \frac{4}{25} = \frac{6}{275}$

$\Rightarrow a + b + 1 = 11 \Rightarrow a + b = 10$

$\Rightarrow a = 4, b = 6$

$\Rightarrow C_2 = 3, C_3 = 5$

$\Rightarrow C_1 = \frac{1}{\beta(3,5)} = 105$

Sol.(39) (A, B, C) A, B are by definition

$$x_1 - x_2 + x_3 \sim N(0, 3)$$

$$\Rightarrow \quad \frac{x_1 - x_2 + x_3}{\sqrt{3}} \sim N(0, 1)$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \sim x^2(4)$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}\left(\frac{x_1 - x_2 + x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}\right) \sim t(4)$$

$$\Rightarrow \quad E\left(\frac{x_1 - x_2 + x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}\right) = 0$$

Sol.(40) (A, C, D) $E[(x-2)^2] = E(x^2 - 4x + 4) = E(x^2) - 4E(x) + 4$

$= \hat{\theta}_1 - 4$ is an unbiased estimator.

$E[(x-2)^3] = E(x^3 - 6x^2 + 12x - 8) = \hat{\theta}_2 - 6\hat{\theta}_1 + 16$ is an unbiased estimator.

Sol.(41) 0.63

$$E(y) = \int_{-\infty}^{\infty} y f_x(x)dx = \int_0^1 e^{-x}.(1)dx = [-e^{-x}]_0^1 = 1 - e^{-1} = 0.63$$

Sol.(42) 0.5

$M_y(t) = e^{2t+2t^2}$ is of normal distribution with $\mu = 2$, $\frac{\sigma^2}{2} = 2$

$\Rightarrow \quad \mu = 2, \sigma^2 = 4$

$$P(x < 100) = P(\log_{10}x < 2) = P(y < 2) = P\left(\frac{y-2}{2} < 0\right) = \Phi(0) = \frac{1}{2} = 0.5$$

Sol.(43) 0.4

$P(A) = 0.26, P(B) = 0.54, P(A \cap B) = 0.2$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.26 + 0.54 - 0.2 = 0.6$

$P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - 0.6 = 0.4$

Sol.(44) 2

$$\text{Matrix of T} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 3 & 2 & -2 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - (R_1 + R_2)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

Sol.(45) 0.167

$f'(x) = \dfrac{2\tan x}{1-\tan^2 x} = \tan 2x$

Length of arc = $\int_{\pi/12}^{\pi/6} \sqrt{1+\tan^2 2x}dx = \int_{\pi/12}^{\pi/6} \sec 2x dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} \sec x dx$

$= \frac{1}{2}\log(\sec x + \tan s)\Big|_{\pi/6}^{\pi/3} = \frac{1}{2}\left[\log\left(2+\sqrt{3}\right) - 109\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\right] = \frac{1}{2}\left[\log\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)\right]$

= 0.167

Sol.(46) 0.24

$P(y = 0, x = 2) = \frac{1}{25}(0 + 4) = \frac{4}{25}$

$P(x = 2) = P(x = 2, y = 0) + P(x = 2, y = 1) + P(x = 2, y = 2)$

$= \frac{1}{25}(4 + 0) + \frac{1}{25}(4 + 1) + \frac{1}{25}(4 + 4) = \frac{1}{25}(17)$

$P(y = 0 \mid x = 2) = \dfrac{P(x=2, y=0)}{P(x=2)} = \dfrac{4/25}{17/25} = \dfrac{4}{17} = 0.24$

Sol.(47) 21

Area = $\int_0^3 (x^2+4)dx = \frac{x^3}{3} + 4x\Big|_0^3 = 9 + 12 = 21$

Sol.(48) 2

$E(x) = \int_0^5 \frac{3x^3}{125}dx = \frac{3}{5}\cdot\frac{5^4}{125} = 3$

$E(x^2) = \int_0^5 \frac{3x^4}{125}dx = \frac{3}{5}\cdot\frac{5^5}{125} = 15$

$Var(x) = E(x^2) - [E(x)]^2 = 15 - 9 = 6$

$\therefore P\left(|x-3| > \sqrt{3}\right) \le \frac{6}{3} = 2$

Sol.(49) 5

$\int_{-2}^{1} [x]^2 dx = \int_{-2}^{-1} [x]^2 dx + \int_{-1}^{0} [x]^2 dx + \int_{0}^{1} [x]^2 dx = \int_{-2}^{-1} (-2)^2 dx + \int_{-1}^{0} (-1)^2 dx + \int_{0}^{1} (0)^2 dx = 4 + 1$

= 5

Sol.(50) 48

$$E(|x|^3) = \frac{1}{4}\int_{-\infty}^{\infty}|x|^3 e^{-|x|/2}dx = \frac{1}{4}.2.\int_0^{\infty}x^3.e^{-x/2}dx = \frac{1}{2}.16.\int_0^{\infty}t^3.e^{-t}dt$$

$$= 8.[(-t^3 - 3t^2 - 6t - 6)e^{-t}]_0^{\infty} = 8.[6] = 48$$

Sol.(51) 6

Normal equation for a is

$\Sigma y = na + b\Sigma x$

$\Rightarrow \quad 50 = 10a + 30b$

$\Rightarrow \quad 5 = a + 3b$

Normal equation for b is

$\Sigma xy = a\Sigma x + b\Sigma x^2$

$\Rightarrow \quad 160 = 30a + 110b$

$\Rightarrow \quad 16 = 3a + 11b$

Solving, we get a = 3.5, b = 0.5

$\therefore$ y = 3.5 + 0.5x

for x = 5, y = 3.5 + (0.5)5 = 6

Sol.(52) 1.4

Characteristic equation of P is $|\lambda I - P| = 0$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 1 & 1 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = 0$$

$\Rightarrow (\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 7\lambda - 5 = 0$

Characteristic equation of P^{-1} is obtained by replacing λ with $\frac{1}{\lambda}$

$$\Rightarrow \frac{1}{\lambda^3} - \frac{7}{\lambda^2} + \frac{7}{\lambda} - 5 = 0$$

$\Rightarrow 5\lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0$

trace of $P^{-1} = \frac{7}{5}$

Sol.(53) 0.5

$$I = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{dx}{e^{\sin x}+1} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{dx}{e^{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}-x\right)}+1} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{dx}{e^{\sin(-x)}+1} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{e^{\sin x}}{e^{\sin x}+1}dx$$

$$2I = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{dx}{e^{\sin x}+1} + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{e^{\sin x}}{e^{\sin x}+1}dx = \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2}\frac{e^{\sin x}+1}{e^{\sin x}+1}dx\right] = \frac{1}{\pi}[\pi] = 1$$

$$\Rightarrow I = \frac{1}{2} = 0.5$$

Sol.(54) 3

$$E(\hat{\theta}_1) = E\left(\frac{x_1+x_8}{2}\right) = \mu$$

$$E(\hat{\theta}_2) = E\left(\frac{x_1+x_8}{4}+\frac{x_2+x_3+...+x_7}{12}\right) = \mu$$

$\therefore$ Both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased.

$$Var(\hat{\theta}_1) = \frac{\sigma^2}{2}$$

$$Var(\hat{\theta}_2) = \frac{1}{16}(2\sigma^2)+\frac{1}{144}(6\sigma^2) = \frac{\sigma^2}{6}$$

$$e(\hat{\theta}_2,\hat{\theta}_1) = \frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)} = \frac{\sigma^2/2}{\sigma^2/6} = 3$$

Sol.(55) 20

$$E(y_1 - y_2) = \frac{1}{11}[11 \times 12.37] - \frac{1}{9}[9 \times 12.37] = 0$$

$$Var(y_1 - y_2) = Var\left[\frac{1}{11}\sum_{i=1}^{11}x_i - \frac{1}{9}\sum_{i=12}^{20}x_i\right] = \frac{1}{121}\left[\sum_{i=1}^{11}Var(x_i)\right]+\frac{1}{81}\left[\sum_{i=12}^{20}Var(x_i)\right]$$

$$= \frac{1}{121} \times 11 \times 99 + \frac{1}{81} \times 9 \times 99 = 9 + 11 = 20$$

$\therefore \dfrac{[(y_1-y_2)-0]^2}{20}$ has 0 x_1^2 distribution.

$$\Rightarrow \alpha = 20$$

Sol.(56) 5

$x \sim N(\mu, \sigma^2)$

$$\Rightarrow f(x;\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}.e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow \ln[f(x;\mu)] = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\therefore \frac{d}{d\mu}\ln[f(x;\mu)] = \frac{(x-\mu)}{\sigma^2}$$

$$\Rightarrow \frac{d^2}{d\mu^2}\ln[f(x;\mu)] = \frac{-1}{\sigma^2}$$

$$\therefore \text{ Fischer information } = -\int_{-\infty}^{\infty} -\frac{1}{\sigma^2} f(x;\mu).dx = \frac{1}{\sigma^2} = \frac{1}{0.2} = 5$$

Sol.(57) 0.4375

$$P(0 \le x \le 4) = \int_0^4 f(x)dx = \int_0^1 \frac{1}{4}dx + \int_1^4 \frac{1}{4x^2}dx = \frac{1}{4} + \frac{1}{4}.\left[-\frac{1}{x}\right]_1^4 = \frac{1}{4} + \frac{1}{4}\left[1 - \frac{1}{4}\right]$$

$$= \frac{1}{4} + \frac{1}{4}.\frac{3}{4} = \frac{7}{16} = 0.4375$$

Sol.(58) 3

$Av = 0$

$$A = \begin{pmatrix} 5 & -2 & -1 \\ -1 & 1 & -1 \\ -1 & -2 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1 + 4R_2} \begin{pmatrix} 5 & -2 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_1 + 5R_2}$$

$$\begin{pmatrix} 5 & -2 & -1 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore 5v_1 - 2v_2 - v_3 = 0$

$3v_2 - 6v_3 = 0 \quad \Rightarrow v_2 = 2v_3$

$\Rightarrow 5v_1 = 5v_3$

$\Rightarrow v_1 = v_3$

$\therefore$ Unit vector is $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$\frac{|v_2|^2 - |v_3|^2}{|v_1^2|} = \frac{\frac{4}{6} - \frac{1}{6}}{\frac{1}{6}} = 3$$

Sol.(59) 2.24

Slope of tangent at P is

$$\frac{dy}{dx} = x^{3/2}$$

at $\left(1,\frac{2}{5}\right)\frac{dy}{dx} = 1$

$\therefore$ equation of tangent is

$$y - \frac{2}{5} = 1(x - 1)$$

$\Rightarrow \quad x - y = \frac{3}{5}$

This meets x-axis at $Q\left(\frac{3}{5},0\right)$

Length of OQ = $\frac{3}{5}$, PQ = $\sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^2} = \frac{2}{5}\sqrt{2}$

OP = $\sqrt{1^2+\left(\frac{2}{5}\right)^2} = \sqrt{\frac{29}{25}} = \frac{\sqrt{29}}{5}$

$\therefore$ OPQO = $\frac{3}{5}+\frac{\sqrt{29}}{5}+\frac{2\sqrt{2}}{5}$ = **2.24**

Sol.(60) 243

$$x^5\frac{dy}{dx} + 5x^4y = -e^{-x}$$

$\Rightarrow \frac{d}{dx}(x^5.y) = -e^{-x}$

$\Rightarrow x^5y = e^{-x} + C$

$\Rightarrow y = \frac{e^{-x}}{x^5}+\frac{C}{x^5}$

$y(1) = 0 \quad \Rightarrow e^{-1} + C = 0$

$\qquad\qquad \Rightarrow C = -e^{-1}$

$y(-1) = \frac{e-e^{-1}}{-1} = -\left[\frac{e^2-1}{e}\right]$

$y(3) = \frac{e^{-3}-e^{-1}}{3^5} = \frac{e^{-3}[1-e^2]}{3^5}$

$\therefore \frac{1}{e^2}\frac{y(-1)}{y(3)} = \frac{1}{e^2}.\frac{1-e^2}{e}.\frac{3^5.e^3}{1-e^2}$ = **243**